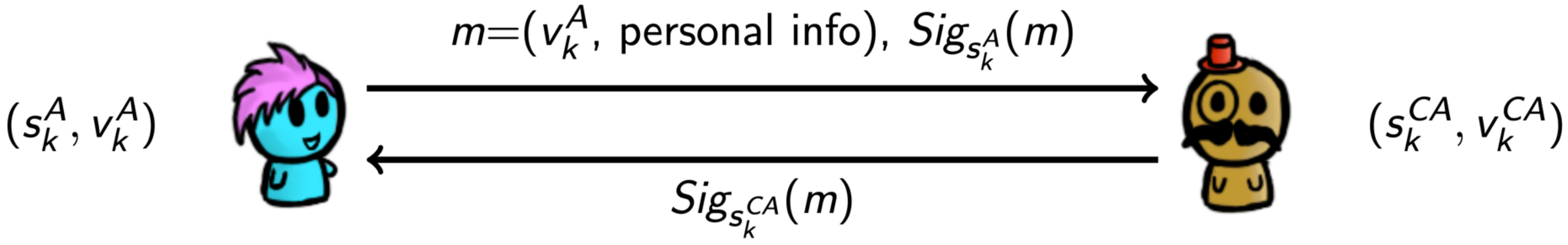
CS459/698 Privacy, Cryptography, Network and Data Security

Discrete Logarithm, Diffie-Hellman, ElGamal

Fall 2024, Tuesday/Thursday 02:30pm-03:50pm

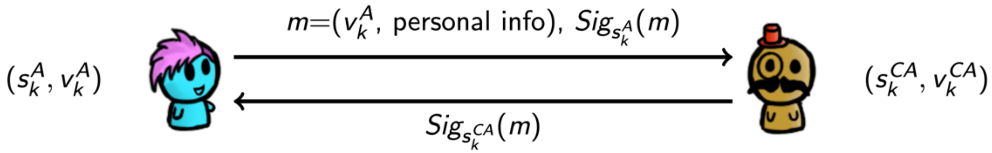
CAs!

Certificate Authorities (CAs)



A **CA** is a trusted third party who keeps a directory of people's (and organizations') verification keys

Certificate Authorities (CAs)

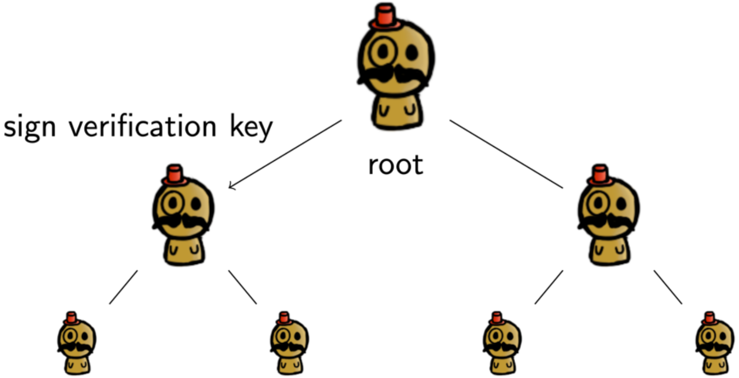


A **CA** is a trusted third party who keeps a directory of people's (and organizations') verification keys

- Alice generates a (s_k^A, v_k^A) key pair, and sends the verification key and personal information, both signed with Alice's signature key, to the CA
- The CA ensures that the personal information and Alice's signature are correct
- The CA generates a certificate consisting of Alice's personal information, as well as her verification key. The entire certificate is signed with the CA's signature key
- https://letsencrypt.org has changed the game. Most web traffic now encrypted. Extended validation certificates (for which CAs charged a lot of money) now not treated differently by browsers.

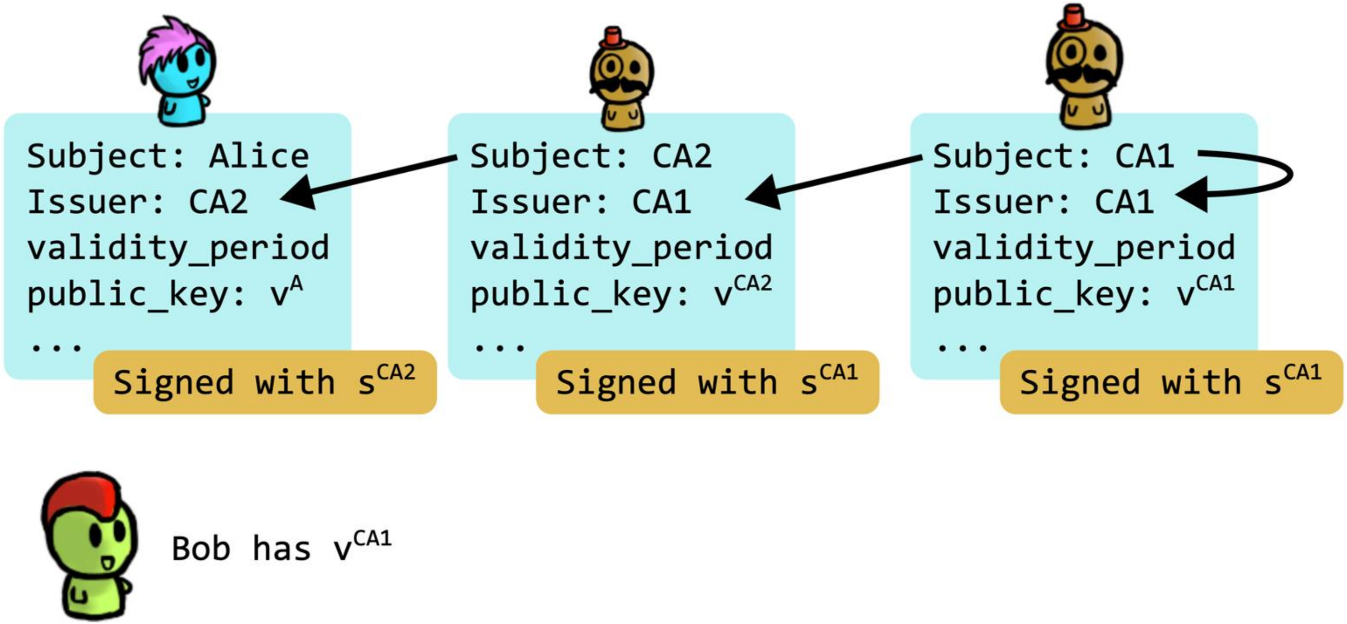
Certificate Authorities (CAs)

- Everyone is assumed to have a copy of the CA’s verification key (v_k^{CA}), so they can verify the signature on the certificate
- There can be multiple levels of certificate authorities; level n CA issues certificates for level n+1 CAs – Public-key infrastructure (PKI)
- Need to have only verification key of root CA to verify the certificate chain



Chain of Certificates

Alice sends Bob the following certificate to prove her identity. Bob can follow the chain of certificates to validate Alice's identity.



Putting it all together

- Secret-key crypto
  - One-time pad
  - Stream ciphers (two-time pad, using nonces)
  - Block ciphers (modes of operation – CBC)
- Public-key crypto
  - Textbook RSA
  - Secret vs. public crypto (speed, key sizes)
  - Hybrid crypto
- Integrity
  - Checksum (usually does not work)
  - Hash functions
- Authentication
  - MACs (repudiation, encrypt-then-MAC)
  - Digital signatures (non-repudiation)
  - Key management
    - Manual keying (SSH)
    - Web of trust (PGP)
    - Certificate authorities (TLS)

The Discrete Logarithm Problem

Given (g,h), find x :

$$g^x = h \bmod p$$

It’s supposed to be hard to find x

I bet we can use that

But don’t forget about me

Groups?

Groups - Sets with specific properties

A **group** is a set of elements (usually numbers) that are related to each other according to some well-defined operations.

- Consider a group of prime order **q**, or Z_q^*
  - This boils down to the set of non-zero integers between 1 and q-1 modulo **q** → A finite group
  - For q = 5, we have group Z_5^* = {1,2,3,4}
  - In this group, operations are carried out mod 5:
    - 3 * 4 = 12 mod 5 = 2
    - 2^3 = 2 * 2 * 2 = 8 mod 5 = 3

Group axioms

To be a group, these sets should respect some axioms

- Closure
- Identity existence
- Associativity
- Inverse existence

- Groups can also be commutative and cyclic (up next)

Let’s take a look at some of these axioms (using multiplication as the operation)

Closure

- For every x,y in the group, x * y is in the group
  - i.e., the multiplication of two group elements falls within the group too

- Example:
  - in Z_5^*, 2* 3 = 6 mod 5 = 1

Identity Existence

- There is an element **e** such that e * x = x * e = x
    - i.e., has an element **e** such that any element times **e** outputs the element itself

- Example:
    - In any Z_q^*, the identity element is 1
    - For Z_5^* : 1 * 3 = 3 mod 5 = 3

Associativity

- For any x, y, z in the group, (x * y) * z = x * (y * z)

- Example:
    - For Z_5^* : (2 * 3) * 4 = 1 * 4 = 2 * (3 * 4) = 2 * 2 = 4

Inverse Existence

- For any **x** in the group, there is a **y** such that x * y = y * x = 1

- Example:
  - For Z_5^* : 2 * 3 = 1 , 3 * 2 = 1 (2 and 3 are inverses)
  - 4 * 4 = 16 mod 5 = 1 (4 is its own inverse)

Abelian Groups

- Abelian groups are groups which are **commutative**
- This means that x * y = y * x for any group elements x and y

- Example:
  - For Z_5^* : 3 * 4 = 2 , 4 * 3 = 2

Cyclic groups

- A group is called **cyclic** if there is at least one element **g** such that its powers (g^1, g^2, g^3, …) mod p span all distinct group elements.
  - **g** is called the "generator" of the group

- Example:
  - For Z_5^*, there are two generators (2 and 3):
    - $2^1 = 2$, $2^2 = 4$, $2^3 = 3$, $2^4 = 1$
    - $3^1 = 3$, $3^2 = 4$, $3^3 = 2$, $3^4 = 1$

Cyclic subgroups

- We can have cyclic **subgroups** within larger finite groups

- Example:
  - Given field F_{607}, we can consider a cyclic subgroup of order p=5 as Z_5^*:

Discrete Logarithm Problem

The Discrete Logarithm Problem

$$h = g^x \text{ , find } x$$

Discrete: we are dealing with integers instead of real numbers

Logarithm: we are looking for the logarithm of **x** base **g**

- e.g., $\log_2 256 = 8$, since $2^8 = 256$

The Discrete Logarithm Problem

Given $(g,h) \in \mathbf{G} \times \mathbf{G}$, find $x \in \mathbf{Z}_q^*$ such that:

$$h = g^x$$

Here, **G** is a multiplicative group of prime order **q**, just like we saw during the examples. (But **q** is thousands of bits long)

Solutions to the Discrete Logarithm Problem?

If there's one solution, there are infinitely many

(thank you Fermat's little theorem and modular arithmetic "wrap-around")

Recall : Let *p* be a prime number and let *a* be any integer. Then:

$$a^{p-1} \equiv \begin{cases} 1 \pmod{p} \text{ if p does not divide a} \\ 0 \pmod{p} \text{ if p does divide a, } p|a \end{cases}$$

How to solve DLP in cyclic groups of prime order?

- Is the group cyclic, finite, and abelian?

Has a generator that spans all elements

Has a limited number of elements

Multiplication is commutative

Baby-step/Giant-step algorithms!!!

How to solve DLP in cycli… grou… of prim… order?

- Is the group cyclic, finite, a…

Has a generator that
spans all ele…

NOTE: for any actual group there may be specialized algorithms which work faster.
e.g. Index Calculus Method

…p/Giant-step algorithms!!!

Oh! … Divi… co… ince the bottleneck is …ing DLP i… cyclic … oups of prime order.

For **generic groups**, the complexity of the Baby-step/giant-step algorithm dominates the time required.

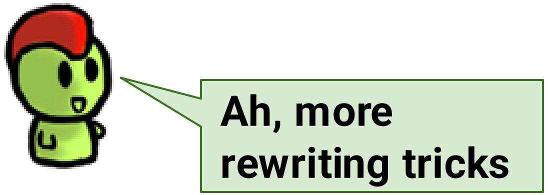
Baby-Step/Giant-Step Algorithm?

- A cyclic group **G** = <g> which has prime order **p**
- $h \in G$, goal: find x (mod p) such that $h = g^x$

- Every element **x** $\in$ G can be written as: **x** = $i + j*\lceil sqrt(p) \rceil$
  - For integers m, i, j satisfying $0 \le i, j \le m$.

Baby-Step/Giant-Step Algorithm?

- A cyclic group **G** = <g> which has prime order **p**
- $h \in G$, goal: find x (mod p) such that $h = g^x$
- Every element **x** $\in$ G can be written as: **x** $= i + j * \lceil \mathrm{sqrt}(p) \rceil$
  - For integers m, i, j satisfying $0 \le i, j \le m$.



Then:

$$h = g^{i + j * \lceil \mathrm{sqrt}(p) \rceil}$$

$$g^i = h \,.\, (g^{-\lceil \mathrm{sqrt}(p) \rceil})^j$$

Baby-Step/Giant-Step Algorithm? Notation.

- **$\log_g x \bmod p$** is obtained by comparing two lists:

$$g^i = h \,.\, (g^{-\lceil sqrt(p) \rceil})^j$$

When we find a coincidence, the equality holds and then $x = i + j * \lceil sqrt(p) \rceil$

Can we divide and conquer?

Baby-step/Giant-Step Algorithm

$g^i = h \,.\, (g^{-\lceil sqrt(p) \rceil})^j$

1. $x = i + j * \lceil sqrt(p) \rceil$

Baby-step/Giant-Step Algorithm

$g^i = h \,.\, (g^{-\lceil sqrt(p) \rceil})^j$

1. $x = i + j * \lceil sqrt(p) \rceil$
2. $0 \leq i, j < \lceil sqrt(p) \rceil$

Since 0≤x≤p, …

$g^i = h \cdot (g^{-\lceil sqrt(p) \rceil})^j$

Baby-step/Giant-Step Algorithm

1. $x = i + j * \lceil sqrt(p) \rceil$
2. $0 \leq i, j < \lceil sqrt(p) \rceil$
3. Baby-step: $g_i \longleftarrow g^i$ for $0 \leq i < \lceil sqrt(p) \rceil$

Let's build some tables!

Baby-step/Giant-Step Algorithm

$g^i = h \,.\, (g^{-\lceil sqrt(p) \rceil})^j$

1. $x = i + j * \lceil sqrt(p) \rceil$
2. $0 \leq i, j < \lceil sqrt(p) \rceil$
3. Baby-step: $g_i \longleftarrow g^i$ for $0 \leq i < \lceil sqrt(p) \rceil$

Produces pairs: (g_i, i)

$g^i = h \cdot (g^{-\lceil sqrt(p) \rceil})^j$

Baby-step/Giant-Step Algorithm

1. $x = i + j * \lceil sqrt(p) \rceil$
2. $0 \le i, j < \lceil sqrt(p) \rceil$
3. Baby-step: $g_i \longleftarrow g^i$ for $0 \le i < \lceil sqrt(p) \rceil$
4. Giant-step: $h_j \longleftarrow h * g^{-j \lceil sqrt(p) \rceil}$, for $0 \le j < \lceil sqrt(p) \rceil$

Produces pairs: (h_j, j)

Baby-step/Giant-Step Algorithm

$g^i = h \cdot (g^{-\lceil sqrt(p) \rceil})^j$

1. $x = i + j * \lceil sqrt(p) \rceil$
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Produces pairs: (h_j, j)

Overall time and space $O(sqrt(p))$

Baby-step/Giant-Step Alg m

1. x = i + j*⌈sqrt(p)⌉

2. 0≤ i, j < ⌈sqrt(p)⌉

3.

4. Gia sqrt(p)⌉

(h_{j},j)

Note: For DLP in group G to be "<u>difficult enough</u>" (e.g., 2^{128} operations), needs prime order subgroup of size greater than 2^{256}

Overall time a space $O(\text{sqrt}(p))$

DLP Example, $182 = 64^x \pmod{607}$

- Consider the subgroup of order 101 (Z_{101}^*) in $\boldsymbol{F}_{607}$, generated by g=64

i	64^i (mod 607)	*i*	" "
0		6	
1		7	
2		8	
3		9	
4		10	
5		-	

Baby-step: $g_i \longleftarrow g^i$ for $0 \leq i < \lceil sqrt(p) \rceil$

g = 64
$\lceil$sqrt(p)$\rceil$ = 11

DLP Example, $182 = 64^x (\mathrm{mod}\ 607)$

i	64^i (mod 607)	*i*	" "
0	1	6	330
1	64	7	482
2	454	8	498
3	527	9	308
4	343	10	288
5	100	-	

Baby-step: $g_i \longleftarrow g^i$ for $0 \le i < \lceil \mathrm{sqrt}(p) \rceil$

$g = 64$

$\lceil \mathrm{sqrt}(p) \rceil = 11$

DLP Example, $182 = 64^x (\bmod 607)$

Giant-step: $h_j \longleftarrow h*g^{-j\,\lceil sqrt(p) \rceil}$

$g = 64$
$\lceil sqrt(p) \rceil = 11$

j	$182 * 64^{-11*j}$ (mod 607)	*j*	
0		6	
1		7	
2		8	
3		9	
4		10	
5		-	

DLP Example, $182 = 64^x \pmod{607}$

Giant-step: $h_j \longleftarrow h * g^{-j\,\lceil sqrt(p) \rceil}$

$g = 64$

$\lceil sqrt(p) \rceil = 11$

j	$182 * 64^{-11*j} \pmod{607}$	*j*	
0	182	6	60
1	143	7	394
2	69	8	483
3	271	9	76
4	343	10	580
5	573	-	

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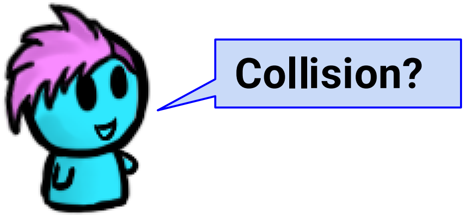

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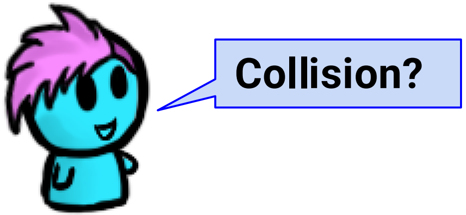

j		*j*	$182 \cdot 64^{-11 \cdot j} \pmod{607}$
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1	143	7	394
2	69	8	483
3	271	9	76
4	343	10	580
5	573	-	

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3	527	9	308
4	**343**	10	288
5	10		

Collision?

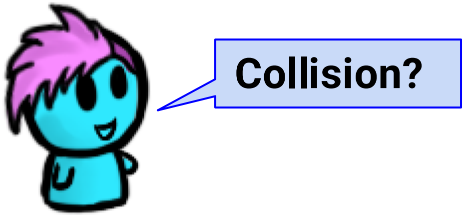
j		j	$182 * 64^{-11*j} \pmod{607}$
0	182	6	60
1	143	7	394
2	69	8	483
3	271	9	76
4	**343**	10	580

Match when **i=4** and **j=4**.

(i is not necessarily equal to j, but it happened on this run ¯_(ツ)_/¯

DLP Example, $182 = 64^x \pmod{607}$

i		*i*	64^i (mod 607)
0	1	6	330
1	64	7	482
2	454	8	498
3	527	9	308
4	343	10	288
5	100		



j		*j*	$182 * 64^{-11*j}$ (mod 607)
0	182	6	60
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So: $x = 4 + 4*11 = 48$.

DLP Example, $182 = 64^x \pmod{607}$

i		i	$64^i \pmod{607}$
0	1	6	330
1	64	7	482
2	454	8	498
3	527	9	308
4	**343**	10	288
5	[illegible]	[illegible]	[illegible]

Collision?

j		j	$182 \cdot 64^{-11 \cdot j} \pmod{607}$
0	182	6	60
1	143	7	394
2	69	8	483
3	271	9	76
[illegible]	[illegible]	[illegible]	[illegible]

Verify: $64^{48} \pmod{607} = 182$

So: $x = 4 + 4 \cdot 11 = 48$.

Diffie-Hellman

Diffie-Hellman Key Exchange

A public-key protocol published in 1976 by Whitfield Diffie and Martin Hellman

Allows two parties that have no prior knowledge of each other to jointly establish a shared secret key over an insecure channel

Key used to encrypt subsequent communications using a symmetric key cipher

Diffie-Hellman Key Exchange

- Used for establishing a <u>shared secret</u> (lacks authentication; we'll see why this is bad)
- Assume as public parameters generator **g** and prime **p**
- Alice (resp. Bob) generates private value **a** (resp. **b**)

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- Assume as public parameters generator **g** and prime **p**
- Alice (resp. Bob) generates private value **a** (resp. **b**)

$A = g^a \bmod p$ →

← $B = g^b \bmod p$

$B^a = (g^b)^a = g^{ba}$

$A^b = (g^a)^b = g^{ab}$

Alice and Bob can derive the same value by exchanging public values and combining them with their private ones!

Diffie-Hellman Key Exchange

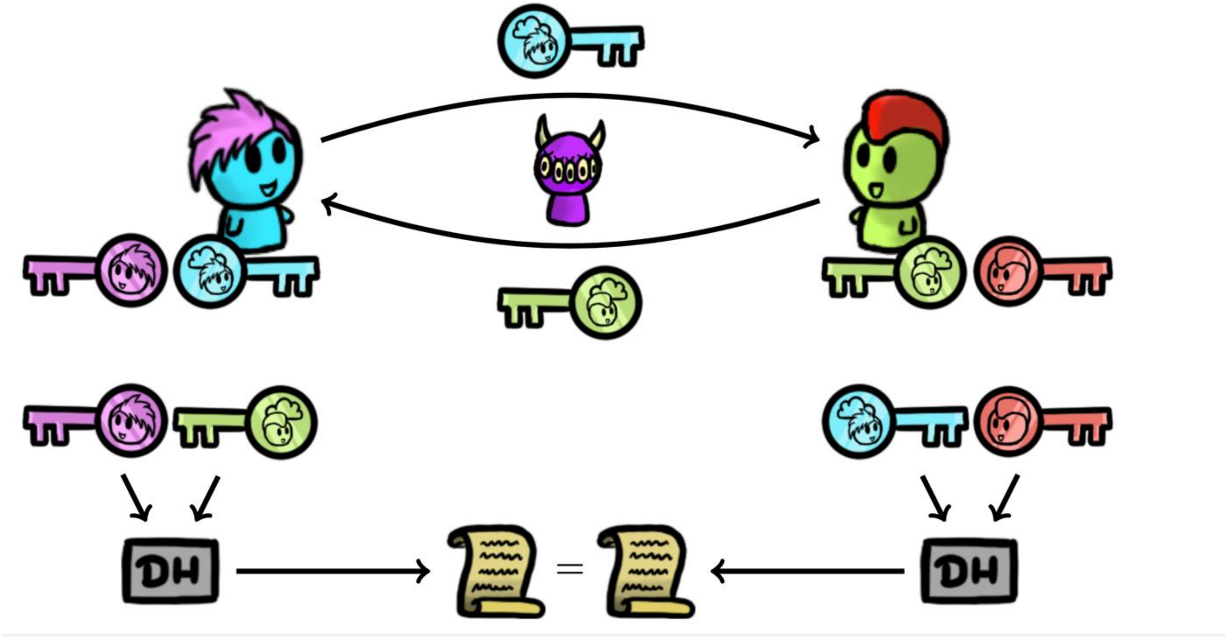
- Used for establishing a <u>shared secret</u> (lacks authentication; we'll see why this is bad)
- Assume as public parameters generator **g** and prime **p**
- Alice (resp. Bob) generates private value **a** (resp. **b**)

$B^a = (g^b)^a = g^{ba}$

$A^b = (g^a)^b = g^{ab}$

Resist keying temptation: the shared value should not <u>immediately</u> be used as a key. g^{ab} is a random element inside a group, but not necessarily a random bit string

Diffie-Hellman Key Exchange – Visualization



Diffie-Hellman relies on the DLP

DH can be broken by recovering the private value **a** from the public value $\mathbf{g^a}$

The adversary <u>must not</u> be able to solve the DLP

The Decisional Diffie-Hellman Problem

Given $\mathbf{g}$, $\mathbf{g}^a$, $\mathbf{g}^b$ distinguish $\mathbf{g}^{ab}$ from random $\mathbf{g}^c$

- An adversary should NOT be able to learn anything about the secret $\mathbf{g}^{ab}$ after observing public values $\mathbf{g}^a$ and $\mathbf{g}^b$
  - Assume $\mathbf{g}^{ab}$ and $\mathbf{g}^c$ occur with the same probability
  - For unknowns a,b, and c.

The Decisional Diffie-Hellman Problem

Given **g**, $\mathbf{g^a}$, $\mathbf{g^b}$ distinguish $\mathbf{g^{ab}}$ from random $\mathbf{g^c}$

- **Challenger** chooses c s.t. c=a*b with Pr=1/2 or c is random 
  - **Goal** of the adversary is to determine whether:

c=a*b **OR** random *c*

DDH Security Game

bit $\longleftarrow$ {0,1}

g $\leftarrow$ G

a,b $\leftarrow$ $\mathbf{Z}_q$

If bit=0 then c $\leftarrow$ $\mathbf{Z}_q$

If bit=1 then c $\leftarrow$ a*b

g^a, g^b, g^c

bit’

Win if bit’=bit

A

$\mathbf{Adv}_{\mathbf{G}}^{\mathbf{DDH}}\mathbf{(A)}$ = 2*|Pr[A wins the DDH game in G]-½|.

The Decisional Diffie-Hellman Problem

Given $\mathbf{g}$, $\mathbf{g}^{a}$, $\mathbf{g}^{b}$ distinguish $\mathbf{g}^{ab}$ from random $\mathbf{g}^{c}$

- An adversary should NOT be able to learn anything about the secret $\mathbf{g}^{ab}$ after observing public values $\mathbf{g}^{a}$ and $\mathbf{g}^{b}$
  - Assume $\mathbf{g}^{ab}$ and $\mathbf{g}^{c}$ occur with the same probability
  - For unknowns a, b, and c.

Useful assumption **beyond** DH key exchange! ⇒ **ElGamal** relies on the DDH assumption

ElGamal

- **1985 by Taher ElGamal**

ElGamal Public Key Cryptosystem

- Let ***p*** be a prime such that the DLP in $(\mathbf{Z}_p^*, .)$ is infeasible
- Let α be a generator in $\mathbf{Z}_p^*$ and **a** a secret value
- $\boldsymbol{Pub}_K = \{(p, \alpha, \beta): \beta \equiv \alpha^a \pmod p)\}$

Public key is p, α, β

- For message **m** and secret random **k** in $\mathbf{Z}_{p-1}$:
  - $e_K(m,k) = (y_1, y_2)$, where $\mathbf{y_1} = \alpha^k \bmod p$ and $\mathbf{y_2} = m\beta^k \bmod p$
- For y_1, y_2 in $\mathbf{Z}_p^*$:
  - $d_K(y_1, y_2) = y_2(y_1{}^a)^{-1} \bmod p$

ElGamal: The Keys

1. Bob picks a “large” prime **p** and a generator ”primitive root” α.
    a. Assume message m is an integer $0 < m < p$
2. Bob picks secret integer **a**
3. Bob computes $\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$

ElGamal: The Keys

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4. Bob’s public key is **(p,** α**, β)**

ElGamal: The Keys

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   a. Assume message m is an integer $0 < m < p$

2. Bob picks secret integer **a**
3. Bob computes $\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$
4. Bob's public key is **(p,** α**, β)** 
5. Bob's private key is **a** 

Bob's Pub_K → **(p, α, β)**

Bob's Priv_K → **a**

$\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$

ElGamal: Encryption

I choose secret integer **k**

ElGamal: Encryption

Bob's Pub_K → $(\mathbf{p}, \boldsymbol{\alpha}, \boldsymbol{\beta})$

Bob's $Priv_K$ → $\mathbf{a}$

$\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$

I choose secret integer **k**

Compute $\mathbf{y_1} \equiv \alpha^k \pmod{p}$

ElGamal: Encryption

Bob's $Pub_K \rightarrow$ **(p, α, β)**

Bob's $Priv_K \rightarrow$ **a**

$\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$

I choose secret integer **k**

Compute $\mathbf{y_1} \equiv \alpha^{\mathbf{k}} \pmod{p}$

Compute $\mathbf{y_2} \equiv \beta^k m \pmod{p}$

Bob's $Pub_K \rightarrow$ **(p, α, β)**

Bob's $Priv_K \rightarrow$ **a**

$\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$

ElGamal: Encryption

I choose secret integer **k**

Compute $\mathbf{y_1} \equiv \alpha^{\mathbf{k}} \pmod{p}$

Compute $\mathbf{y_2} \equiv \beta^k m \pmod{p}$

Send $\mathbf{y_1}$ and $\mathbf{y_2}$ to Bob

Bob's Pub_K → **(p, α, β)**

Bob's $Priv_K$ → **a**

$\beta \equiv \alpha^a \pmod{p}$

ElGamal: Decryption

I choose secret integer **k**

Compute $\mathbf{y_1} \equiv \alpha^{k} \pmod{p}$

Compute $\mathbf{y_2} \equiv \beta^k m \pmod{p}$

Compute $\mathbf{y_1 y_2^{-a}} \equiv m \pmod{p}$

Send $\mathbf{y_1}$ and $\mathbf{y_2}$ to Bob

Bob's Pub_K → **(p, α, β)**
Bob's $Priv_K$ → **a**
$\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$

ElGamal: Decryption

I choose secret integer **k**

Compute $\mathbf{y_1} \equiv \alpha^{\mathbf{k}} \pmod{p}$

Compute $\mathbf{y_2} \equiv \beta^k m \pmod{p}$

Compute $\mathbf{y_2 y_1^{-a}} \equiv m \pmod{p}$

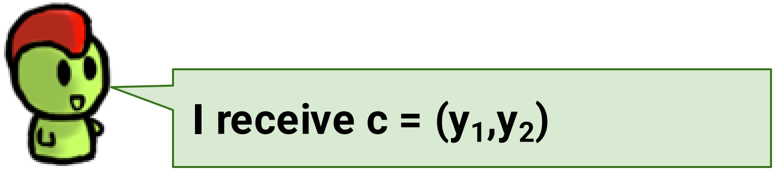
Send $\mathbf{y_1}$ and $\mathbf{y_2}$ to Bob

Bob can <u>decrypt</u> since:

$\mathbf{y_2 y_1^{-a}} \equiv \beta^k m (\alpha^k)^{-a} \equiv m \pmod{p}$

ElGamal Informal Summary

- The plaintext m is “hidden” by multiplying it by β^k to get y_2



ElGamal Informal Summary

- The plaintext m is "hidden" by multiplying it by β^k to get y_2
- The ciphertext includes α^k so that Bob can compute β^k from α^k (because Bob knows a)

I receive c = (y_1,y_2)

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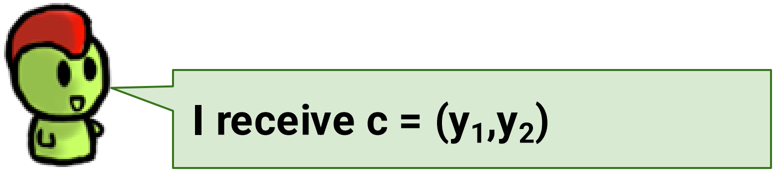

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Bob's $Priv_K$ → **a** = 765

$\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$

Example

- Let **p**=2579 , $\boldsymbol{\alpha}$ = 2 , **β** = 2^{765} mod 2579 = 949

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- $\mathbf{y_1}$ = 2^{853} mod 2579 = 435
- $\mathbf{y_2}$=$1299*949^{853}$ mod 2579 = 2396

Send y_1, y_2 to Bob

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- Bob now has $\mathbf{y_1}$ and $\mathbf{y_2}$
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- $m = 2396 * 435^{-765} \bmod 2759 = 1299$

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Insecure if the adversary can compute $\mathbf{a} = \log_\alpha \beta$

To be secure, DLP must be infeasible in $\boldsymbol{Z}_p{}^*$

Nice! That's the plaintext I wanted to send.

ElGamal…Encrypt. “Small” Calculation Day

- $(p, \alpha, \beta) = (809, 256, 498)$
- $a = 68$
- $k = 89$
- $m=100$

Determine $c = y_1, y_2$.

Submit c and a short description of your computation.

Network Security - Next class