

# CS459/698

# Privacy, Cryptography, Network and Data Security

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Homomorphic Encryption

Fall 2025, Tuesday/Thursday 8:30-9:50am

# What is Homomorphic Encryption?

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- **Definition:** Homomorphic encryption is a cryptographic technique that allows computations to be performed on encrypted data without requiring decryption.
- Raw data can remain fully encrypted while it's being processed, manipulated, and run through various algorithms.

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- Raw data can remain fully encrypted while it's being processed, manipulated, and run through various algorithms.
- Idealized in 1978, fully realized in 2009 by Craig Gentry



Fully Homomorphic Encryption Using Ideal Lattices

Craig Gentry  
Stanford University and IBM Watson  
cgentry@cs.stanford.edu

# Homomorphic Encryption for Dummies

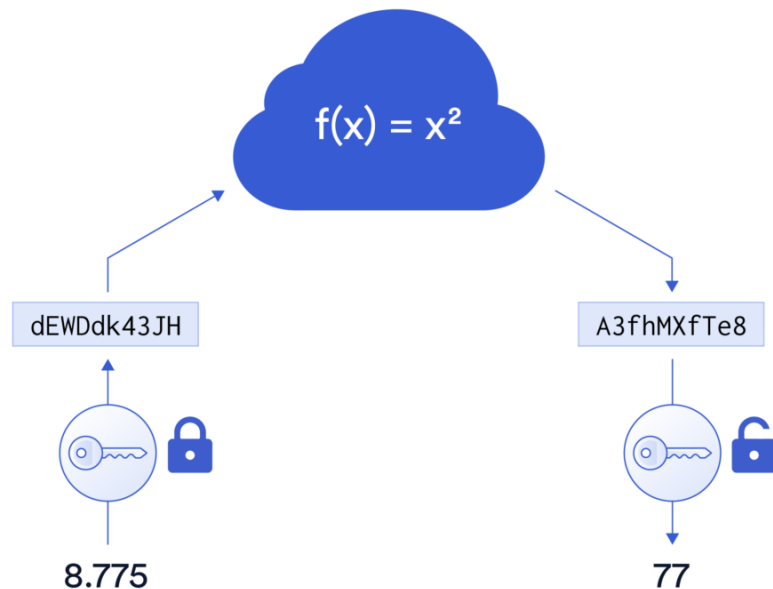
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*“Anybody can come and they can stick their hands inside the gloves and manipulate what’s inside the locked box. They can’t pull it out, but they can manipulate it; they can process it... Then they finish and the person with the secret key has to come and open it up—and only they can extract the finished product out of there.”*

**-- Craig Gentry**

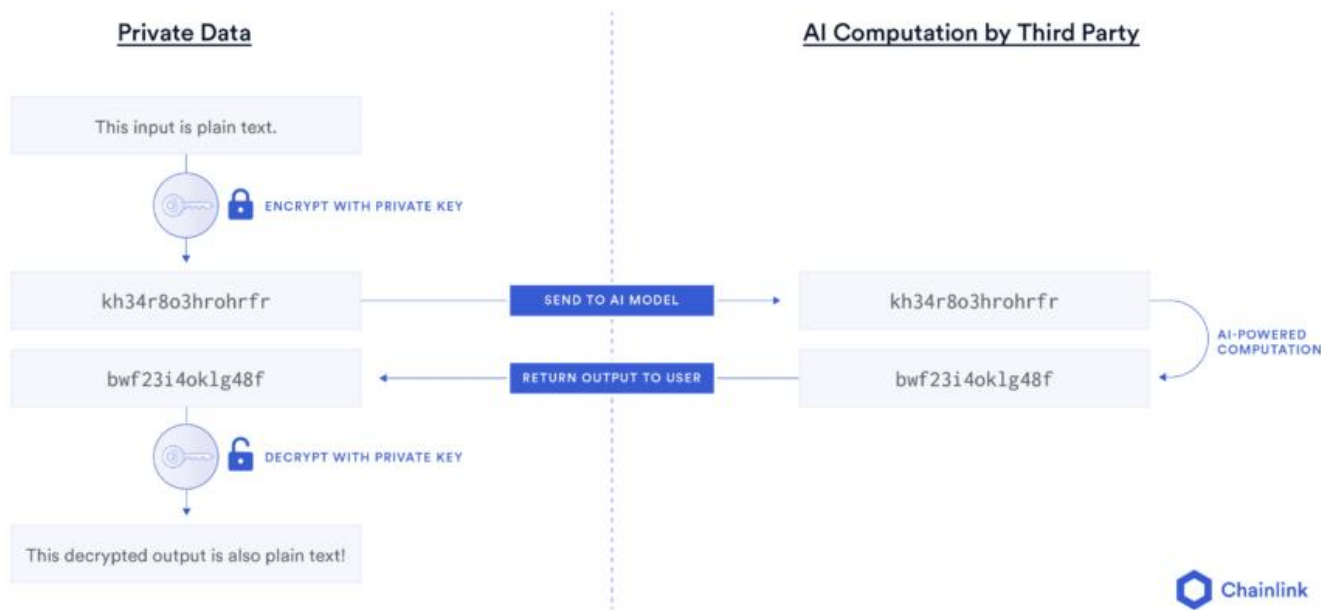
<https://www.youtube.com/watch?v=pXb39wj5ShI>

# Computing on Ciphertexts (Simple Math)



<https://chain.link/education-hub/homomorphic-encryption>

# Computing on Ciphertexts (More sophisticated math)



<https://chain.link/education-hub/homomorphic-encryption>

# Homomorphic Encryption in the Wild



<https://dualitytech.com/blog/homomorphic-encryption-making-it-real/>



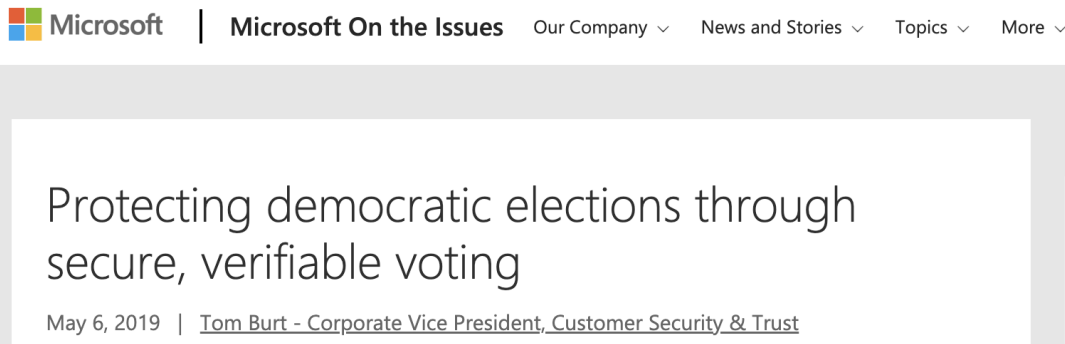
# Homomorphic Encryption in the Wild

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- Used as a tool in many real-world scenarios:
  - <https://www.ibm.com/security/services/homomorphic-encryption>
  - <https://www.statcan.gc.ca/en/data-science/network/homomorphic-encryption>
  - <https://www.statcan.gc.ca/en/data-science/network/statistical-analysis-homomorphic-encryption>
  - <https://www.intel.com/content/www/us/en/developer/tools/homomorphic-encryption/overview.html>
  - <https://www.microsoft.com/en-us/research/project/microsoft-seal/>

# E.g., Homomorphic Encryption for Secure Voting

- Microsoft's ElectionGuard



## What does ElectionGuard do?

ElectionGuard is a way of checking election results are accurate, and that votes have not been altered, suppressed or tampered with in any way. Individual voters can see that their vote has been accurately recorded, and their choice has been correctly added to the final tally. Anyone who wishes to monitor the election can check all votes have been correctly tallied to produce an accurate and fair result.

# So what is this all about?

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# Homomorphic Encryption

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Consider the following:

Two ciphertexts use the same key,  $\mathbf{c} = E_K(\mathbf{x})$ ,  $\mathbf{d} = E_K(\mathbf{y})$

Let  $\mathbf{f}()$  be a function that operates over plaintext  $\mathbf{x}$  and  $\mathbf{y}$

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**Goal:** the existence of a function  $\mathbf{g}()$  such that

$$\mathbf{g}(\mathbf{c}, \mathbf{d}) = E_K(\mathbf{f}(\mathbf{x}, \mathbf{y}))$$

# Homomorphic Encryption

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$$\mathbf{g}(\mathbf{c}, \mathbf{d}) = E_K(\mathbf{f}(\mathbf{x}, \mathbf{y}))$$

**$\mathbf{g}()$  is a homomorphic function on the ciphertexts  $\mathbf{c}, \mathbf{d}$**

# Partial versus Fully Homomorphic Encryption

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The function on the plaintexts is:

...either multiplication or addition **but not both**.

**Partial HE**

...either multiplication or addition, **or both**

**Fully HE**

# Four Shades of Homomorphic Encryption

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Homomorphic Encryption Types				
	Partially	Somewhat	Leveled Fully	Fully
Rating	Simple	Intermediate	Advanced	Most advanced
Computations	Addition or multiplication	Addition and/or multiplication	Complex but limited	Complex and unlimited
Use cases	Sum or product	Basic statistical analysis	AI/ML, MPC	AI/ML, MPC

<https://chain.link/education-hub/homomorphic-encryption>



# Four Shades of Homomorphic Encryption

Only useful for **simpler** operations. Relatively **efficient**.

RSA, ElGamal, Paillier

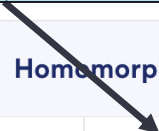
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# Four Shades of Homomorphic Encryption

# of operations that can be performed is **bounded** and the accuracy of the computation may **degrade** as more operations are performed.

BGN



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# Four Shades of Homomorphic Encryption

BFV, CKKS, TFHE (leveled)

Can perform an **arbitrary** # of computations on encrypted data, if it has a pre-defined set of computations **specified ahead of time**.

Homomorphic Encryption Types				
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# Four Shades of Homomorphic Encryption

BGV, DGHV,  
TFHE (w/bootstrapping)

Enables **any** # of computations to be performed on encrypted data **without a predefined sequence or limit**. Computationally **expensive**.

Homomorphic Encryption Types				
	Partially	Somewhat	Leveled Fully	Fully
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# A partial homomorphic encryption scheme based on El Gamal

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# Recap: ElGamal Public Key Cryptosystem

- Let  $p$  be a prime such that the DLP in  $(\mathbf{Z}_p^*, \cdot)$  is infeasible
- Let  $\alpha$  be a generator in  $\mathbf{Z}_p^*$  and  $a$  a secret value
- **PubK** =  $\{(p, \alpha, \beta): \beta \equiv \alpha^a \pmod{p}\}$
- For message  $m$  and secret random  $k$  in  $\mathbf{Z}_{p-1}$ :
  - $e_K(m, k) = (y_1, y_2)$ , where  $y_1 = \alpha^k \pmod{p}$  and  $y_2 = m\beta^k \pmod{p}$
- For  $y_1, y_2$  in  $\mathbf{Z}_p^*$ :
  - $d_K(y_1, y_2) = y_2(y_1^a)^{-1} \pmod{p}$



Public key is  $p, \alpha, \beta$



# Consider Multiplicative HE

Bob's  $\text{Pub}_K \rightarrow (p, \alpha, \beta)$

Bob's  $\text{Priv}_K \rightarrow a$

$$y_1 \equiv \alpha^k \pmod{p}$$

$$y_2 \equiv m \beta^k \pmod{p}$$

$$\beta \equiv \alpha^a \pmod{p}$$



$$f(x, y) = x \cdot y$$

**Private key:  $a$ , public key:  $\alpha^a$**

**Instead of  $k$ , choose  $r$  and  $s$**

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**Goal:** show how the multiplication of ciphertexts corresponds to the multiplication of plaintexts.



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$$f(x, y) = x \cdot y$$

**Private key:  $a$ , public key:  $\alpha^a$**

**Instead of  $k$ , choose  $r$  and  $s$**

$$c = (\alpha^r, x \alpha^{ra})$$

$$d = (\alpha^s, y \alpha^{sa})$$

**Idea:** Create ciphertexts for the two different plaintexts

# Consider Multiplicative HE

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**Private key:  $a$ , public key:  $\alpha^a$**

**Instead of  $k$ , choose  $r$  and  $s$**

$$c = (\alpha^r, x \alpha^{ra})$$

$$d = (\alpha^s, y \alpha^{sa})$$

**Idea:** combine ciphertexts of two different plaintexts

$$\begin{aligned} g(c, d) &= (\alpha^r \alpha^s, x \alpha^{ra} y \alpha^{sa}) \\ &= (\alpha^{r+s}, xy \alpha^{a(r+s)}) \end{aligned}$$



# Consider Multiplicative HE



$$f(x, y) = x \cdot y$$

**Private key:  $a$ , public key:  $\alpha^a$**

**Instead of  $k$ , choose  $r$  and  $s$**

$$c = (\alpha^r, \mathbf{x} \alpha^{ra})$$

$$d = (\alpha^s, \mathbf{y} \alpha^{sa})$$

**Idea:** decrypt the combined ciphertext

$$g(c, d) = (\alpha^{r+s}, \mathbf{xy} \alpha^{a(r+s)})$$

$$d_K(\alpha^{r+s}, \mathbf{xy} \alpha^{a(r+s)}) = \mathbf{xy} \alpha^{a(r+s)} / \alpha^{a(r+s)} = \mathbf{xy}$$

# Consider Additive HE

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**Multiplicative:** The math of ElGamal ensures that multiplying the encrypted values corresponds to multiplying the original values.

**Additive:** Here, we no longer have the same nice properties of how exponents play together.

- **“Crazy” idea:** Something like  $g(E_K(a^x), E_K(a^y)) = E_K(a^{x+y})$  could work
  - But we would need to break the discrete log of  $a^{x+y}$  to retrieve the sum
    - Only really works for small  $x$  and  $y$

# The Paillier Partially Homomorphic Encryption Scheme

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# Paillier's Encryption Scheme

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- Proposed by Pascal Pailler in 1999
- The Paillier cryptosystem is a public-key cryptosystem known for its **additive** homomorphic properties.
- The security of the Paillier cryptosystem is based on the difficulty of the **composite residuosity class problem**
  - This problem involves determining whether a given number is an  $n$ -th residue modulo  $n^2$  for a composite  $n$ .

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  - Determining whether a given number is an  $n$ -th residue modulo  $n^2$  for a composite  $n$ .



**Public-Key Cryptosystems Based on Composite Degree Residuosity Classes**  
Published in J. Stern, Ed., *Advances in Cryptology – EUROCRYPT '99*,  
1999, Lecture Notes in Computer Science, pp. 223–238,  
Springer-Verlag, 1999.  
P. Paillier, 1,2  
Cryptography Department  
INRIA, Rocquencourt, France

# Paillier's Encryption Scheme

---

- Let  $p, q$  be two large primes;  $N = pq$
- Ciphertexts are mod  $N^2$
- Choose  $r$ ; plaintext  $m \pmod{p}$  is encrypted as  $g^m r^N \pmod{N^2}$

g is a generator





# Paillier's Encryption Scheme

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From the **product of ciphertexts** to **addition of plaintexts**

- Multiply encryption of  $m_1$  and  $m_2$ :

$$\begin{aligned} E(m_1, r_1) \cdot E(m_2, r_2) \bmod N^2 &= \\ g^{m_1} \cdot g^{m_2} \cdot r_1^N \cdot r_2^N \bmod N^2 &= \\ g^{m_1+m_2} \cdot (r_1 \cdot r_2)^N \bmod N^2 \end{aligned}$$

# Paillier's Encryption Scheme

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- If factorization of  $\mathbf{N}$  is known, breaking the DL is efficient  
 $\Rightarrow$  Efficient additive HE, even for large numbers

# Paillier's Encryption Scheme

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$$g^{m_1+m_2} \cdot (r_1 \cdot r_2)^N \bmod N^2$$

- If factorization of  $N$  is known, breaking the DL is efficient  
⇒ Efficient additive HE, even for large numbers

$$D(E(m_1, r_1) \cdot E(m_2, r_2) \bmod n^2) = m_1 + m_2 \bmod n.$$



Simplica Numara!

# DGHV: A Fully Homomorphic Encryption Scheme

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# Fully Homomorphic Encryption (FHE)

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- Many schemes now, usually abbreviated by the first letters of the last names of the authors
- Different security assumptions (not factoring or discrete log)
  - Lattice problems: Learning with errors, ...

## Examples:

- First construction by Gentry in 2009
- E.g. FV, BGV, or DGHV (not used in practice)

# The **DGHV** Fully Homomorphic Encryption Scheme

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- FHE scheme whose security is based on the difficulty of the **approximate greatest common divisor** (AGCD) problem.
  - Finding the greatest common divisor of a set of integers that are close to multiples of a secret integer.

## Fully Homomorphic Encryption over the Integers

Marten van Dijk  
MIT

Craig Gentry  
IBM Research

Shai Halevi  
IBM Research

Vinod Vaikuntanathan  
IBM Research

June 8, 2010

<https://medium.com/@j248360/explaining-the-dghv-encryption-scheme-1acb6cd74dd6>

<https://www.esat.kuleuven.be/cosic/blog/co6gc-homomorphic-encryption-part-1-computing-with-secrets/>

<https://github.com/coron/fhe>

# Consider Simplified DGHV (not used in practice)

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- $m \in \{0, 1\}$
- Secret key: prime  $p$

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- $m \in \{0, 1\}$
- Secret key: prime  $p$
- **Encryption**
  - Choose  $q, r$  such that  $r < p$   $\rightarrow r$  is random noise
  - $c = qp + 2r + m$



# Consider Simplified DGHV (not used in practice)

---

- $m \in \{0, 1\}$
- Secret key: prime  $p$
- **Encryption**
  - Choose  $q, r$  such that  $r < p$   $\rightarrow r$  is random noise
  - $c = qp + 2r + m$
- **Decryption**
  - $m = c \bmod 2 \oplus (\lfloor c/p \rfloor \bmod 2)$

# Computing with Simplified DGHV

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- **Ciphertexts**

- $c_1 = q_1p + 2r_1 + m_1$

- $c_2 = q_2p + 2r_2 + m_2$

# Computing with Simplified DGHV

---

- **Ciphertexts**

- $c_1 = q_1p + 2r_1 + m_1$

- $c_2 = q_2p + 2r_2 + m_2$

- **Addition**

- $c_1 + c_2 = (q_1 + q_2)p + 2(r_1 + r_2) + m_1 + m_2$



Note that noise grows **linearly**

# Computing with Simplified DGHV

- **Ciphertexts**

- $c_1 = q_1 p + 2r_1 + m_1$

- $c_2 = q_2 p + 2r_2 + m_2$

- **Addition**

- $c_1 + c_2 = (q_1 + q_2)p + 2(r_1 + r_2) + m_1 + m_2$

- **Multiplication**

- $c_1 \cdot c_2 = q'p + 2r' + m_1 m_2$

- $r' = 2r_1 r_2 + r_1 m_2 + r_2 m_1$

- $q' = q_1 q_2 p + q_1 m_2 + q_2 m_1$



Note the increased growth of the noise.  
(no longer linear). One gets a new  
ciphertext with noise **roughly twice larger**  
than in the original ciphertexts  $c_1$  and  $c_2$ .

# The bootstrapping problem in FHE

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# Bootstrapping... in Fully HE Schemes

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- If  $r > p/2 \Rightarrow$  decryption fails on DGHV
  - Also a problem for other schemes.
- If the noise **grows too much**, it can **corrupt** the encrypted data and make it unusable
- Each operation **increases the noise**, so one must **control** this growth

# Bootstrapping... in Fully HE Schemes

- To obtain a FHE scheme, (i.e. unlimited addition and multiplication on ciphertexts), one must **reduce** the amount of noise in a ciphertext
- **Bootstrapping** is a procedure that reduces noise
  - E.g., in DGHV, homomorphically compute the decryption function for obtaining an encryption of  $m$  with lower noise
  - Still, bootstrapping is **slow** in most fully HE schemes
  - Thus, with fully HE, aim to **avoid** subsequent multiplications



# Practical FHE Schemes

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- **FV, BGV, BFV, CKKS**

- Lattice-based encryption schemes
- Encrypt vectors (usually as polynomials)

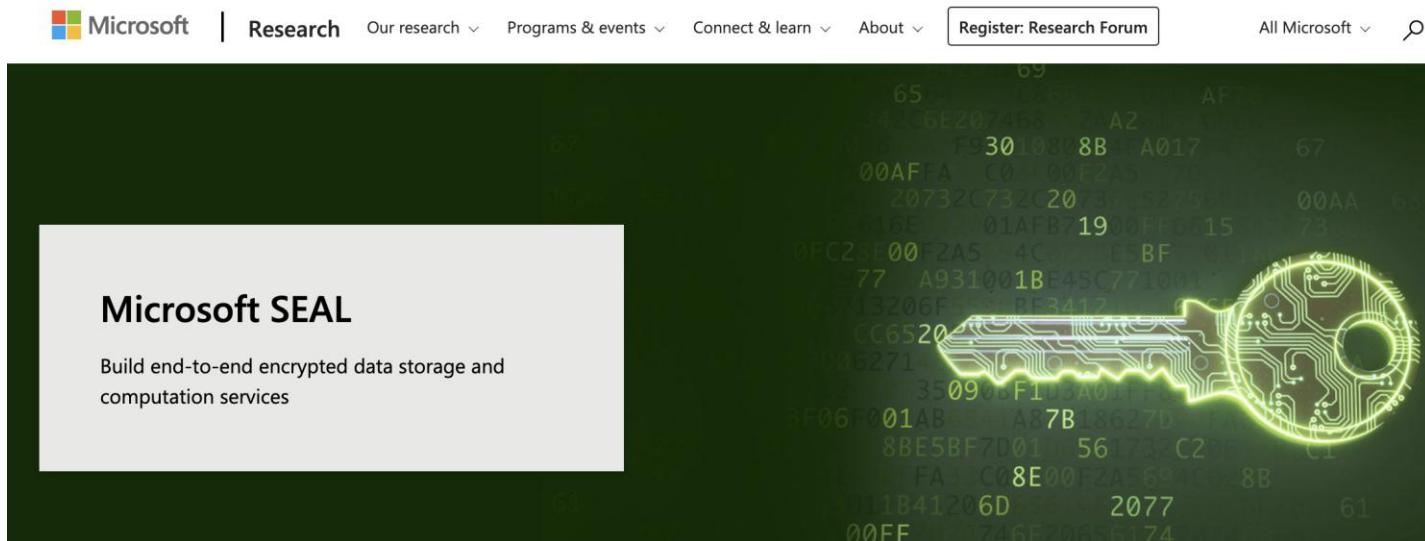
- **TFHE**

- Fully HE over the Torus
- Usually encrypts bits
- Very fast bootstrapping (frequently performed)
- <https://tfhe.github.io/tfhe/>



# Try it... on your own 😊

- Download Microsoft's SEAL library and hack away!
  - <https://www.microsoft.com/en-us/research/project/microsoft-seal/>



# Quick announcement (again 😊)

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- **Student Course Perceptions** (<https://perceptions.uwaterloo.ca/>)

- Close on Dec 2, 11:59pm
- Did you like it? Did you hate it? Let me know!

