CS489/689
Privacy, Cryptography, Network and Data Security

Homomorphic encryption, MPC, and PSI

Winter 2023, Tuesday/Thursday 8:30-9:50am

Computing on Ciphertexts

Consider the following:

Two ciphertexts use the same key, $c_1 = E_K(\mathbf{a})$, $c_2 = E_K(\mathbf{b})$

Let **f()** be a function that operates over plaintext **a** and **b**

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g() is a **homomorphic function** on the ciphertexts c, d, …

Homomorphic Encryption in the World

- Used as a tool in many business scenarios:
  - https://www.ibm.com/security/services/homomorphic-encryption
  - https://www.statcan.gc.ca/en/data-science/network/homomorphic-encryption
  - https://www.microsoft.com/en-us/research/project/microsoft-seal/

Partial versus Fully Homomorphic Encryption

The function on the plaintexts is:

…either multiplication or addition **but not both.**

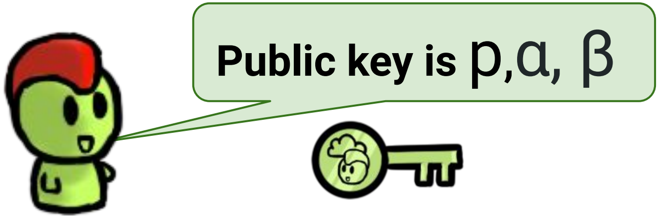
Partial HE

…either multiplication or addition **both…or even xor**

Fully HE

Recall ElGamal Public Key Cryptosystem

- Let p be a prime such that the DLP in $(\mathbf{Z}_p^*, \cdot)$ is infeasible
- Let $\alpha \in \mathbf{Z}_p^*$ be a primitive element
- Let $\mathcal{P} = \mathbf{Z}_p^*$, $\mathcal{C} = \mathbf{Z}_p^* \times \mathbf{Z}_p^*$ and…
- $\mathcal{K} = \{(p, \alpha, a, \beta) : \beta \equiv \alpha^a \pmod{p}\}$
- For a secret random number k in $\mathbf{Z}_{p-1}$ define:
  - $e_K(x,k) = (y_1, y_2)$, where $y_1 = \alpha^k \bmod p$ and $y_2 = x\beta^k \bmod p$
- For y_1, y_2 in $\mathbf{Z}_p^*$, define $d_K(y_1, y_2) = y_2(y_1^{\,a})^{-1} \bmod p$



Consider Multiplicative HE

$f(a, b) = a \cdot b$

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g(c, d):

- $e_1 = c_1 \cdot d_1 = g^r g^s = g^{r+s}$
- $e_2 = c_2 \cdot d_2 = ab\, g^{rx} g^{sx} = ab\, g^{x(r+s)}$

Consider Additive HE

- Multiplicative HE: Idea: encrypt a, b as g^a and g^b, respectively
- $g(E_A(g^a), E_A(g^b)) = E_A(g^{a+b})$
- Need to break discrete logarithm of g^{a+b}
  - Only works for small a, b

Note of Caution!

- Paillier's
- Simplified DGHV
- …

Some mathematical details have been omitted from the remainder of the lecture

Paillier's Encryption Scheme

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⇒ Efficient additive HE for large numbers

$$D(E(m_1,r_1) \cdot E(m_2,r_2) \bmod n^2) = m_1 + m_2 \bmod n$$

Product to addition

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$$D(E(m_1, r_1) \cdot g^{m_2} \bmod n^2) = m_1 + m_2 \bmod n$$

Fully HE

- Need to encrypt message m in the base, then both operations work
- Many schemes now, usually abbreviated by the first letters of the last names of the authors
- Different security assumptions (not factoring or discrete log)
  - Lattice problems: Learning with errors, …

Examples:

- First construction by Gentry in 2009
- E.g. FV, BGV, or DGHV (not used in practice)

Consider Simplified DGHV (not used in practice)

- $m \in \{0, 1\}$
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  - $c = qp + 2r + m$
- **Decryption**
  - $m = c \bmod 2 \oplus (\lfloor c/p \rfloor \bmod 2)$

Computing with Simplified DGHV

- Ciphertexts
  - $c_1 = q_1 p + 2r_1 + a$
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- Addition
  - $c_1 + c_2 = (q_1+q_2)p + 2(r_1+r_2) + a + b$
- Multiplication
  - $c_1 \cdot c_2 = ??? = q'p + 2r' + ab$
  - $r' = 2r_1 r_2 + r_1 b + r_2 a$
    - Note the increase in length!!

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- $r > p/2 \Rightarrow$ decryption fails
- Multiplication quickly increases noise (doubles the length)
- Bootstrapping is a procedure that reduces the noise to its initial length
  - Bootstrapping is slow in most fully HE schemes
    - DGHV does not have bootstrapping
  - When using fully HE, it is therefore important to reduce the number of subsequent multiplications

Practical Used FHE

- FV, BGV, BFV, CKKS
  - Lattice-based encryption schemes
  - Encrypt vectors (usually as polynomials)
- TFHE
  - Fully HE over the Torus
  - Usually encrypts bits
  - Very fast bootstrapping (frequently performed)

Try it…

- Download Microsoft's SEAL library
  - https://www.microsoft.com/en-us/research/project/microsoft-seal/
- Create a key
- Encrypt two 8 bit numbers bit-wise using batch encoding (allows rotation)
- Perform comparison, for each position: If prefix is equal and bits are different, output 1 if bit of first number is 1; else output 0
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Try it…on your own.

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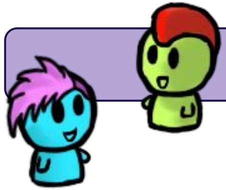
Multi-Party Computation (MPC)

What is Multi-Party Computation?



1) **At least two parties**

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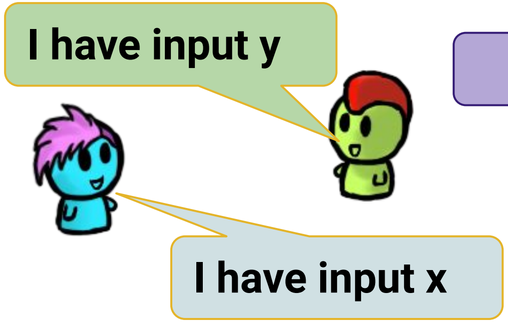

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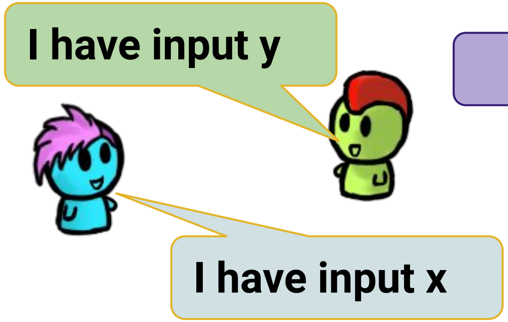

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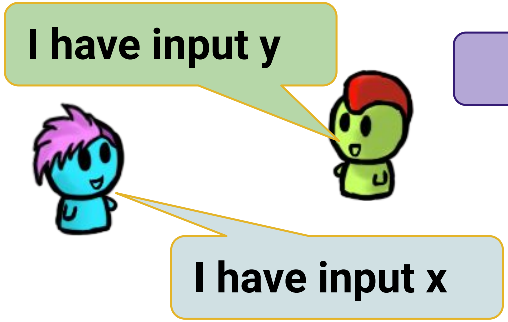

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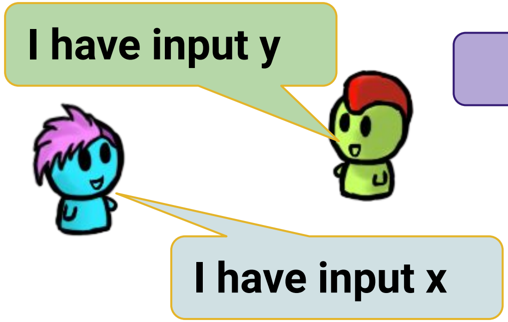

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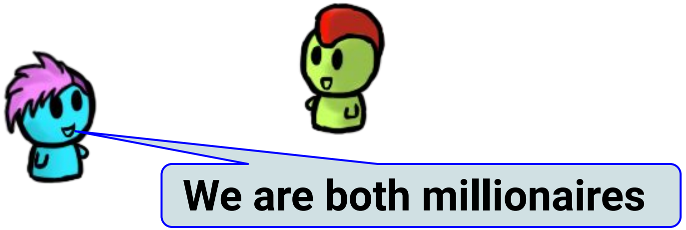
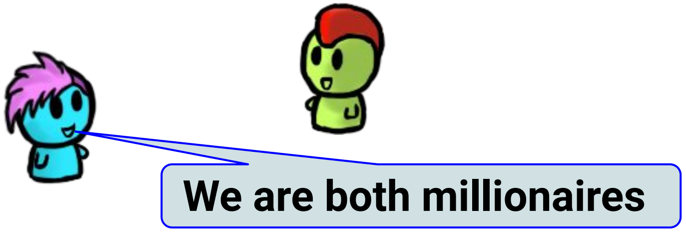
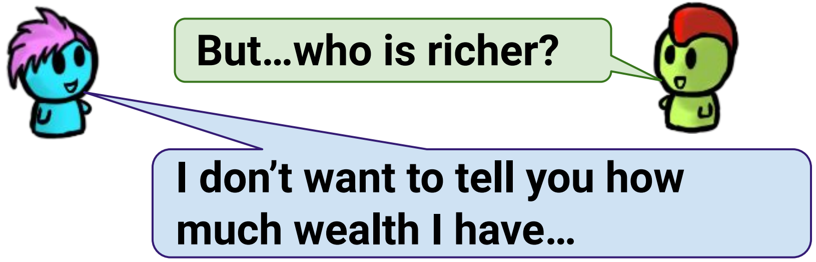
Critical: Secret inputs, public outputs (to at least one party)

Toy Example, Basically "Millionaire's Problem"



Toy Example, “The Millionaire's Problem”





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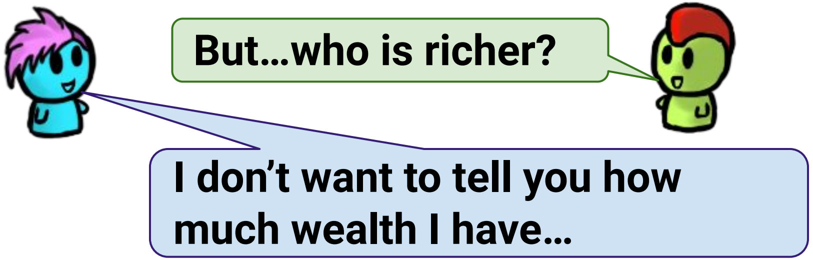

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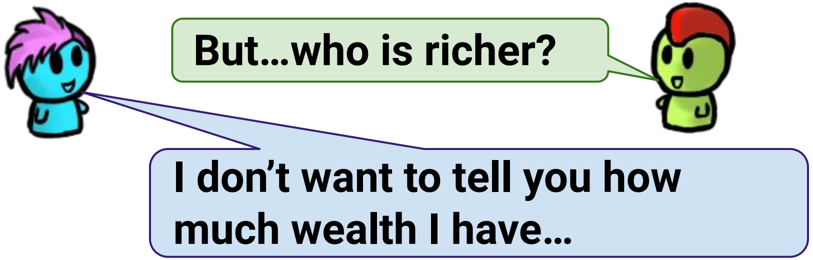

Q: how can Bob and Alice determine who is richer?

A: A multi-party computation to compute the f: x < y

Fun Facts:

- “Yao’s millionaires’ problem” (Andrew C. Yao, Turing Award 2000)

Exponential Solution

Let E_a be Alice's public key. Alice has i billions, Bob has j millions, such that $1 < i, j < 10$.

1. Bob picks a random N-bit integer, and computes privately the value of $E_a(x)$; call the result k.

2. Bob sends Alice the number $k - j + 1$

3. Alice computes privately the values of $y_u = D_a(k - j + u)$ for $u = 1, 2, \ldots, 10$.

Exponential Solution Con't

4. Alice generates a random prime p of N/2 bits, and computes the values $\mathbf{z_u = y_u \ (mod\ p)}$ for all u; if all z_u differ by at least 2 in the mod p sense, stop; otherwise generates another random prime and repeat the process until all z_u differ by at least 2; let p, z_u denote this final set of numbers;

5. Alice sends the prime p and the following 10 numbers to B: $z_1, z_2, \ldots, z_i$ followed by $z_i + 1, z_{i+1} + 1, \ldots, z_{10} + 1$; the above numbers should be interpreted in the mod p sense.

Exponential Solution More Con’t

6. Bob looks at the j-th number (not counting p) sent from Alice, and decides that $i \geq j$ if it is equal to x mod p, and $i < j$ otherwise.

7. Bob tells Alice what the conclusion is.

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Short A: Other than lies…no.

“Real-World” Example

I want to analyse sentence x (NLP)

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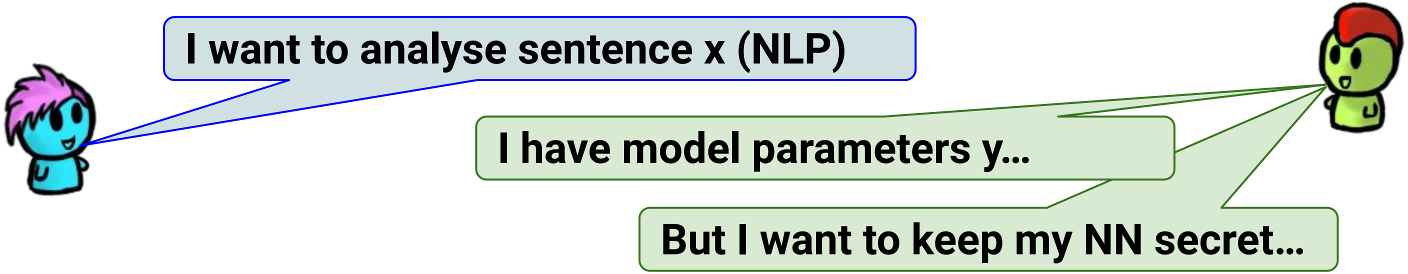

“Real-World” Example

I want to analyse sentence x (NLP)

I have model parameters y…

But I want to keep my NN secret…

Require: A function f over public parameters, but secret architecture

Goal: A MPC for f(x, y) such that only Alice learns the analysis of her sentence and Alice does not learn the NN

“Types” of MPC: Participant Set

Two-party

Multi-Party

MPC Server Model

- Assume n >> 3 clients with an input
  - Example collect statistics about emoji usage in texting

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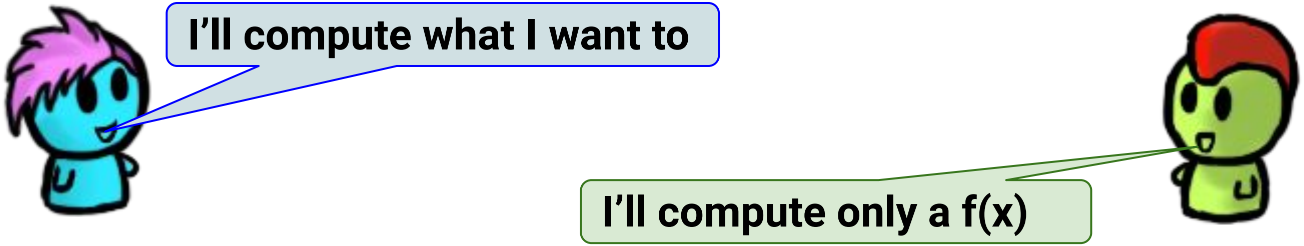
- The servers perform multi-party computation
  - Decrypt input
  - Compute f

“Types” of MPC: Functionality



Generic

Specific

Generic / Specific Functions, in more words

- Specific functions:

A multi-party computation protocol that can only be used for **a specific function f**

- Generic functions:

A multi-party computation protocol that can be used for **“any” function f**

“Types” of MPC: Security

I’m beyond curious

Active

I’m just curious

Passive

Passive Security

- Passive security
  (also called security against **semi-honest adversaries**)

  Each party **follows the protocol** but keeps a record of all messages and after the protocol is over, **tries to infer additional information** about the other parties’ inputs

Active Security

- Active security

  (also called security against **malicious adversaries**)

  Each party **may arbitrarily deviate from the protocol**. Either the protocol computes f or the protocol is aborted.

Relation of Passive and Active Security

- Passive security is a **prerequisite** for active security

Q: What does a prerequisite here mean?

Relation of Passive and Active Security

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Q: Suggestions?

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  - Adding additional protocol steps proving the correct computation of each message

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